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ESTIMATES ON INITIAL COEFFICIENT BOUNDS OF SUBCLASSES OF BI-UNIVALENT FUNCTIONS DEFINED BY GENERALIZED DIFFERENTIAL OPERATOR

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Abstract: In this present paper, we introduce two new subclasses $\mathcal{H}_{\Sigma}^{k}(\alpha, \delta, \lambda, \mu, \sigma)$ and $\mathcal{H}_{\Sigma}^{k}(\beta, \delta, \lambda, \mu, \sigma)$ of normalized analytic bi-univalent functions defined in the open unit disk and associated with generalized differential operator. Further, we obtain bounds for the second and third Taylor- Maclaurin coefficients of the functions of these subclasses. Also, we obtain some consequences of results obtained.

Keywords and Phrases: Analytic function, Univalent function, Bi-Univalent function, Taylor-Maclaurin series.

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1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of analytic functions f defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by conditions $f(0) = 0$ and $f'(0) = 1$. The series expansion of $f \in \mathcal{A}$ is given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbb{U}. \tag{1.1}$$

Let S denote the subclass of $\mathcal{A}$ containing all univalent functions in $\mathbb{U}$. A function $f \in S$ is said to be starlike of order α $(0 \leq \alpha < 1)$ if $\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha$. A function $f \in S$ is said to be convex of order α $(0 \leq \alpha < 1)$ if $\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$. The classes of starlike functions of order α and convex functions of order α are denoted by $S^*(\alpha)$ and $\mathcal{K}(\alpha)$. Each function $f \in S$ is invertible for some part of unit disk $\mathbb{U}$. In fact, the Koebe one quarter Theorem [1] ensures that, f^{-1} exists at least on $\{z \in \mathbb{C} : |z| < 1/4\}$. Thus every $f \in S$ has an inverse f^{-1} which is defined by

$$f^{-1}(f(z)) = z, \quad z \in \mathbb{U}$$

and

$$f(f^{-1}(w)) = w, \quad |w| < r(f), \quad r(f) \geq 1/4$$

where,

$$f^{-1}(w) = w - a_2 w^2 + (2a_2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denotes the class of bi-univalent functions. For brief history of the class Σ, see Srivastava [20].

Bi-univalent functions serve as essential objects of study in complex analysis, offering insights into the behavior and structure of analytic functions. The study of bi-univalent functions is particularly rich due to their connections with various geometric properties. Bounds on the coefficients of the Taylor series expansion of such functions are indicative of their geometric behavior and analytic properties. Thus, coefficient estimates play a crucial role in analyzing the geometric behavior of any function. Consequently, numerous investigators obtained coefficient bounds for various subclasses of class Σ.

Lewin [12] initiated this journey in 1967 by introducing the class Σ and demonstrating that $|a_2| < 1.51$. Subclasses of the bi-univalent functions of class Σ, similar to the well-known subclasses $S^*(\alpha)$ and $\mathcal{K}(\alpha)$, were introduced by Branan and Taha

[21]. Additionally, a subclass of strongly bi-starlike functions of order α $(0 < \alpha \leq 1)$ and denoted by S^{*}_{Σ} was introduced by Branan and Taha [3].

In [7], Frasin defined the subclass $S(\alpha, \mu, \sigma)$ of analytic functions f satisfying the following condition

$$\Re\left\{\frac{(\mu-\sigma)zf'(z)}{f(\mu z)-f(\sigma z)}\right\} > \alpha$$

for some $0 \leq \alpha < 1$, $\mu, \sigma \in \mathbb{C}$ with $|\mu| \leq 1, |\sigma| \leq 1; \mu \neq \sigma$ and for all $z \in \mathbb{U}$. We also denote by $T(\alpha, \mu, \sigma)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ such that $zf'(z) \in S(\alpha, \mu, \sigma)$. The class $S(\alpha, 1, \sigma)$ was introduced and studied by Owa et. al. [15]. When $\sigma = -1$, the class $S(\alpha, 1, -1) = S_{\mu}(\alpha)$ was introduced by Sakaguchi [18] function of order α (see also [16]), where as $S_{\mu}(0) = S_{\mu}$ is the class of starlike functions with respect to symmetrical points in $\mathbb{U}$. Also, note that $S(\alpha, 1, 0) = S^{*}(\alpha)$ and $T(\alpha, 1, 0) = \mathcal{K}(\alpha)$.

In [5], Darus *et.al.* defined the differential operator $D^{k}_{\lambda,\delta}f(z)$ defined as follows. Let $f \in \mathcal{A}$. The operator $D^{k}_{\lambda,\delta} : \mathcal{A} \to \mathcal{A}$ given by,

$$D^{k}_{\lambda,\delta}f(z) = z + \sum_{n=2}^{\infty}(1+(n-1)\lambda)^{k}C(\delta,n)a_{n}z^{n}, \qquad k \in \mathbb{N}_{0}, \lambda, \delta \geq 0 \tag{1.3}$$

where,

$$C(\delta, n) = \binom{n+\delta-1}{\delta} = \frac{\Gamma(n+\delta)}{\Gamma(n)\Gamma(\delta+1)}.$$

Let us consider,

$$\begin{aligned}
D^{k+1}_{\lambda,\delta}f(z) - (1-\lambda)D^{k}_{\lambda,\delta}f(z) &= \left[z + \sum_{n=2}^{\infty}(1+(n-1)\lambda)^{k+1}C(\delta,n)a_{n}z^{n}\right] \\
&\quad - (1-\lambda)\left[z + \sum_{n=2}^{\infty}(1+(n-1)\lambda)^{k}C(\delta,n)a_{n}z^{n}\right] \\
&= z + \sum_{n=2}^{\infty}(1+(n-1)\lambda)^{k}C(\delta,n)[(1+(n-1)\lambda)-(1-\lambda)]a_{n}z^{n} \\
&= \lambda z(D^{k}_{\lambda,\delta}f(z))'.
\end{aligned}$$

Thus, we have

$$\lambda z(D^{k}_{\lambda,\delta}f(z))' = D^{k+1}_{\lambda,\delta}f(z) - (1-\lambda)D^{k}_{\lambda,\delta}f(z). \tag{1.4}$$

Here, we observe that if $\delta = 0$, we get differential operator defined by Al-Oboudi [1], for $k = 0$ we obtain Ruscheweyh differential operator [17] and for $\lambda = 1$ and $\delta = 0$, we get Salagean differential operator [19].

In this paper, results are obtained by using the following Lemma.

Lemma 1. [1] *If $\mathbb{P} \in \mathcal{P}$ then $|c_n| \leq 2$, $(n \in \mathbb{N})$, where, $\mathcal{P}$ is the family of all functions $\mathbb{P}$, analytic in $\mathbb{U}$, for which $\Re\{\mathbb{P}(z)\} > 0$, $(z \in \mathbb{U})$, where $\mathbb{P}(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$.*

We assume throughout this paper that $\lambda, \delta \geq 0; k \in \mathbb{N}_0; \mu, \sigma \in \mathbb{C}$ with $|\mu|, |\sigma| \leq 1; \mu \neq \sigma$.

In [7], Frasin defined the subclass $S(\alpha, \mu, \sigma)$ which is generalization of $S_\mu(\alpha)$ introduced by Sakaguchi [18] and the functions in this class are called asSakaguchi function of order α. The study of the Sakaguchi function and its properties is significant in understanding the behavior of univalent functions, particularly in the context of geometric function theory. Motivated by earlier works of Frasin [7], Srivastava [20] and Frasin and Aouf [9], Lewin [12] and Branan and Taha [21] and Darus et.al. [5] (See also [4, 8, 13, 10, 11, 14, 22]), we have defined two new subclasses $\mathcal{H}_\Sigma^k(\alpha, \delta, \lambda, \mu, \sigma)$ and $\mathcal{H}_\Sigma^k(\beta, \delta, \lambda, \mu, \sigma)$ by using generalized differential operator and estimated the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in each of these bi-univalent classes.

2. Coefficient bounds for the functions in the class $\mathcal{H}_\Sigma^k(\alpha, \delta, \lambda, \mu, \sigma)$

Definition 1. *A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{H}_\Sigma^k(\alpha, \delta, \lambda, \mu, \sigma)$ if the following conditions are satisfied:*

$$f \in \Sigma \text{ and } \left|\arg\left(\frac{(\mu-\sigma)z(D_{\lambda,\delta}^k f(z))'}{D_{\lambda,\delta}^k f(\mu z) - D_{\lambda,\delta}^k f(\sigma z)}\right)\right| < \frac{\alpha\pi}{2}; \quad (0 < \alpha \leq 1, z \in \mathbb{U}) \quad (2.1)$$

and

$$\left|\arg\left(\frac{(\mu-\sigma)w(D_{\lambda,\delta}^k g(w))'}{D_{\lambda,\delta}^k g(\mu w) - D_{\lambda,\delta}^k(\sigma w)}\right)\right| < \frac{\alpha\pi}{2}; \quad (0 < \alpha \leq 1, w \in \mathbb{U}) \quad (2.2)$$

where the function g is given by (1.2).

Now we begin to find the estimates on the coefficients $|a_2|, |a_3|$ for the function in the class $\mathcal{H}_\Sigma^k(\alpha, \delta, \lambda, \mu, \sigma)$.

Theorem 2.1. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma^k(\alpha, \delta, \lambda, \mu, \sigma)$.*

Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{\begin{array}{c}|2\alpha(1+\lambda)^{2k}(\delta+1)^2(\mu^2+2\mu\sigma+\sigma^2-2\mu-2\sigma)+\alpha(1+2\lambda)^k(\delta+1)(\delta+2)\\ (3-\mu^2-\mu\sigma-\sigma^2)+(1-\alpha)(1+\lambda)^{2k}(\delta+1)^2(2-\mu-\sigma)^2|\end{array}}} \tag{2.3}$$

$$|a_3| \leq \frac{4\alpha^2}{(1+\lambda)^{2k}(\delta+1)^2|(2-\mu-\sigma)^2|}+\frac{4\alpha}{(1+2\lambda)^k(\delta+1)(\delta+2)|(3-\mu^2-\mu\sigma-\sigma^2)|}. \tag{2.4}$$

Proof. From (2.1) and (2.2), we have

$$\left(\frac{(\mu-\sigma)z(D^k_{\lambda,\delta}f(z))'}{D^k_{\lambda,\delta}f(\mu z)-D^k_{\lambda,\delta}f(\sigma z)}\right)=[p(z)]^\alpha \tag{2.5}$$

and

$$\left(\frac{(\mu-\sigma)w(D^k_{\lambda,\delta}g(w))'}{D^k_{\lambda,\delta}g(\mu w)-D^k_{\lambda,\delta}g(\sigma w)}\right)=[q(w)]^\alpha \tag{2.6}$$

where $p(z)$ and $q(w)$ in $\mathcal{P}$ and represented as

$$p(z)=1+c_1z+c_2z^2+c_3z^3+\cdots \tag{2.7}$$

and

$$q(w)=1+d_1w+d_2w^2+d_3w^3+\cdots. \tag{2.8}$$

Using (1.3) and (1.4) in (2.5) and (2.6), we have the following relations:

$$(1+\lambda)^k(\delta+1)(2-\mu-\sigma)a_2=\alpha c_1 \tag{2.9}$$

$$\begin{aligned}&(1+2\lambda)^k\frac{(\delta+1)(\delta+2)}{2}(3-\mu^2-\mu\sigma-\sigma^2)a_3+\\ &\quad(1+\lambda)^{2k}(\delta+1)^2(\mu^2+2\mu\sigma+\sigma^2-2\mu-2\sigma)a_2^2=\alpha c_2+\frac{\alpha(\alpha-1)}{2}c_1^2\end{aligned} \tag{2.10}$$

and

$$-(1+\lambda)^k(\delta+1)(2-\mu-\sigma)a_2=\alpha d_1 \tag{2.11}$$

$$\begin{aligned}&-(1+2\lambda)^k\frac{(\delta+1)(\delta+2)}{2}(3-\mu^2-\mu\sigma-\sigma^2)a_3+[(1+\lambda)^{2k}(\delta+1)^2(\mu^2+2\mu\sigma+y^2\\ &\quad-2\mu-2\sigma)+(1+2\lambda)^k(\delta+1)(\delta+2)(3-\mu^2-\mu\sigma-\sigma^2)]a_2^2=\alpha d_2+\frac{\alpha(\alpha-1)}{2}d_1^2.\end{aligned} \tag{2.12}$$

From (2.9) and (2.11), we get

$$c_1 = -d_1 \tag{2.13}$$

and

$$2(1+\lambda)^{2k}(\delta+1)^2 a_2^2(2-\mu-\sigma)^2 = \alpha^2(c_1^2+d_1^2). \tag{2.14}$$

Now adding (2.10) and (2.12), we obtain that

$$\begin{aligned}&[2(1+\lambda)^{2k}(\delta+1)^2(\mu^2+2\mu\sigma+\sigma^2-2\mu-2\sigma)+(1+2\lambda)^k(\delta+1)(\delta+2)\\&\quad(3-\mu^2-\mu\sigma-\sigma^2)]a_2^2 = \alpha(c_2+d_2)+\frac{\alpha(\alpha-1)}{2}(c_1^2+d_1^2).\end{aligned} \tag{2.15}$$

From (2.14) and (2.15), we have

$$a_2^2 = \frac{\alpha^2(c_2+d_2)}{\begin{array}{c}2\alpha(1+\lambda)^{2k}(\delta+1)^2(\mu^2+2\mu\sigma+\sigma^2-2\mu-2\sigma)+\alpha(1+2\lambda)^k(\delta+1)(\delta+2)\\(3-\mu^2-\mu\sigma-\sigma^2)+(1-\alpha)(1+\lambda)^{2k}(\delta+1)^2(2-\mu-\sigma)^2\end{array}}$$

Applying Lemma 1 for the coefficients c_2 and d_2, we get

$$|a_2| \le \frac{2\alpha}{\sqrt{\begin{array}{c}|2\alpha(1+\lambda)^{2k}(\delta+1)^2(\mu^2+2\mu\sigma+\sigma^2-2\mu-2\sigma)+\alpha(1+2\lambda)^k(\delta+1)(\delta+2)\\(3-\mu^2-\mu\sigma-\sigma^2)+(1-\alpha)(1+\lambda)^{2k}(\delta+1)^2(2-\mu-\sigma)^2|\end{array}}}$$

Now we find bound on $|a_3|$, by subtracting (2.12) from (2.10), we obtain

$$\begin{aligned}&(1+2\lambda)^{2k}(\delta+1)(\delta+2)(3-\mu^2-\mu\sigma-\sigma^2)a_3-(1+2\lambda)^{2k}(\delta+1)(\delta+2)\\&\quad(3-\mu^2-\mu\sigma-\sigma^2)a_2^2 = \alpha(c_2-d_2)+\frac{\alpha(\alpha-1)}{2}(c_1^2-d_1^2).\end{aligned} \tag{2.16}$$

From (2.13) and (2.14), (2.16) becomes

$$a_3 = \frac{\alpha^2(c_1^2+d_1^2)}{2(1+\lambda)^{2k}(\delta+1)^2(2-\mu-\sigma)^2}+\frac{\alpha(c_2-d_2)}{(1+2\lambda)^k(\delta+1)(\delta+2)(3-\mu^2-\mu\sigma-\sigma^2)}.$$

Now applying Lemma 1 for the coefficients c_1, c_2, d_1 and d_2, we have

$$|a_3| \le \frac{4\alpha^2}{(1+\lambda)^{2k}(\delta+1)^2|(2-\mu-\sigma)^2|}+\frac{4\alpha}{(1+2\lambda)^k(\delta+1)(\delta+2)|(3-\mu^2-\mu\sigma-\sigma^2)|}.$$

Which is desired estimate of $|a_3|$ as given in Theorem 2.1.

Putting $k=0$ in Theorem 2.1, we get the following Corollary.

Corollary 1. *If $f(z)$ given by (1.1) be in the class $\mathcal{H}^0_\Sigma(\alpha,\delta,\mu,\sigma), 0<\alpha\le 1$ then*

$$|a_2| \le \frac{2\alpha}{\sqrt{\begin{array}{l}|2\alpha(\delta+1)^2(\mu^2+2\mu\sigma+\sigma^2-2\mu-2\sigma)+\alpha(\delta+1)(\delta+2)(3-\mu^2-\mu\sigma-\sigma^2)\\ \quad +(1-\alpha)(\delta+1)^2(2-\mu-\sigma)^2|\end{array}}}$$

and

$$|a_3| \le \frac{4\alpha^2}{(\delta+1)^2|(2-\mu-\sigma)^2|}+\frac{4\alpha}{(\delta+1)(\delta+2)|(3-\mu^2-\mu\sigma-\sigma^2)|}.$$

Putting $\delta=0$ in Corollary 1, we get the following Corollary.

Corollary 2. [2] *If $f(z)$ given by (1.1) be in the class $\mathcal{H}^0_\Sigma(\alpha,\mu,\sigma),\ 0<\alpha\le 1$ then*

$$|a_2| \le \frac{2\alpha}{\sqrt{|2\alpha(\mu\sigma+3-2\mu-2\sigma)+(1-\alpha)(2-\mu-\sigma)^2|}}$$

and

$$|a_3| \le \frac{4\alpha^2}{|(2-\mu-\sigma)^2|}+\frac{2\alpha}{|(3-\mu^2-\mu\sigma-\sigma^2)|}.$$

Putting $\mu=1$ and $\sigma=-1$ in Corollary 2, we get the following Corollary.

Corollary 3. [2] *If $f(z)$ given by (1.1) be in the class $\mathcal{H}^0_\Sigma(\alpha,1,-1),\ 0<\alpha\le 1$ then*

$$|a_2|\le\alpha \quad \text{and} \quad |a_3|\le\alpha(\alpha+1).$$

Putting $\mu=1$ and $\sigma=0$ in Corollary 2, we obtain well-known the class $S^*_\Sigma[\alpha]$ of strongly bi-starlike functions of order α and we get the following Corollary.

Corollary 4. [4] *If $f(z)$ given by (1.1) be in the class $S^*_\Sigma[\alpha],\ 0<\alpha\le 1$. Then*

$$|a_2|\le\frac{2\alpha}{\sqrt{\alpha+1}} \quad \text{and} \quad |a_3|\le\alpha(4\alpha+1).$$

3. Coefficient bounds for the functions in the class $\mathcal{H}^k_\Sigma(\beta,\delta,\lambda,\mu,\sigma)$

Definition 2. *A function $f(z)$ given by (1.1) is said to be in class $\mathcal{H}^k_\Sigma(\beta,\delta,\lambda,\mu,\sigma)$ if the following conditions are satisfied.*

$$f\in\Sigma \quad \textit{and} \quad \Re\left\{\frac{(\mu-\sigma)z(D^k_{\lambda,\delta}f(z))'}{D^k_{\lambda,\delta}f(\mu z)-D^k_{\lambda,\delta}f(\sigma z)}\right\}>\beta; \qquad (0\le\beta<1, z\in\mathbb{U}) \quad (3.1)$$

and

$$\Re\left\{\frac{(\mu-\sigma)w(D^k_{\lambda,\delta}g(w))'}{D^k_{\lambda,\delta}g(\mu w)-D^k_{\lambda,\delta}g(\sigma w)}\right\}>\beta; \qquad (0\le\beta<1, z\in\mathbb{U}), \tag{3.2}$$

where the function g is given by (1.2).

Theorem 3.1. *Let function $f(z)$ given by (1.1) be in class $\mathcal{H}^k_\Sigma(\beta,\delta,\lambda,\mu,\sigma)$. Then*

$$|a_2|\le\sqrt{\frac{4(1-\beta)}{\begin{array}{r}2(1+\lambda)^{2k}(\delta+1)^2|\mu^2+2\mu\sigma+\sigma^2-2\mu-2\sigma|+\\(1+2\lambda)^k(\delta+1)(\delta+2)|3-\mu^2-\mu\sigma-\sigma^2|\end{array}}} \tag{3.3}$$

and

$$|a_3|\le\frac{4(1-\beta)}{(1+2\lambda)^k(\delta+1)(\delta+2)|3-\mu^2-\mu\sigma-\sigma^2|}+\frac{4(1-\beta)^2}{(1+\lambda)^{2k}(\delta+1)^2|(2-\mu-\sigma)^2|}. \tag{3.4}$$

Proof. It follows from (3.1) and (3.2), that there exists p and $q\in\mathcal{P}$ such that

$$\frac{(\mu-\sigma)z(D^k_{\lambda,\delta}f(z))'}{D^k_{\lambda,\delta}f(\mu z)-D^k_{\lambda,\delta}f(\sigma z)}=\beta+(1-\beta)p(z) \tag{3.5}$$

and

$$\frac{(\mu-\sigma)w(D^k_{\lambda,\delta}g(w))'}{D^k_{\lambda,\delta}g(\mu w)-D^k_{\lambda,\delta}g(\sigma w)}=\beta+(1-\beta)q(w) \tag{3.6}$$

where $p(z)$ and $q(w)$ are given by (2.7) and (2.8).
Using (1.3) and (1.4) in (3.5) and (3.6), we obtain the following relations:

$$(1+\lambda)^k(\delta+1)(2-\mu-\sigma)a_2=(1-\beta)c_1 \tag{3.7}$$

$$\begin{aligned}&(1+2\lambda)^k\frac{(\delta+1)(\delta+2)}{2}(3-\mu^2-\mu\sigma-\sigma^2)a_3\\&\quad+(1+\lambda)^{2k}(\delta+1)^2(\mu^2+2\mu\sigma+\sigma^2-2\mu-2\sigma)a_2^2=(1-\beta)c_2\end{aligned} \tag{3.8}$$

and

$$-(1+\lambda)^k(\delta+1)(2-\mu-\sigma)a_2=(1-\beta)d_1 \tag{3.9}$$

$$\begin{aligned}&[(1+2\lambda)^k(\delta+1)(\delta+2)(3-\mu^2-\mu\sigma-\sigma^2)+(1+\lambda)^{2k}(\delta+1)^2(\mu^2+2\mu\sigma+\sigma^2\\&\quad-2\mu-2\sigma)]a_2^2-(3-\mu^2-\mu\sigma-\sigma^2)(1+2\lambda)^k\frac{(\delta+1)(\delta+2)}{2}a_3=(1-\beta)d_2.\end{aligned} \tag{3.10}$$

From (3.7) and (3.9), we have

$$c_1 = -d_1 \tag{3.11}$$

and

$$2(1+\lambda)^{2k}(\delta+1)^2(2-\mu-\sigma)^2 a_2^2 = (1-\beta)^2(c_1^2+d_1^2). \tag{3.12}$$

Now by adding (3.8) and (3.10), we deduce that

$$a_2^2 = \frac{(1-\beta)(c_2+d_2)}{\begin{bmatrix}2(1+\lambda)^{2k}(\delta+1)^2(\mu^2+2\mu\sigma+\sigma^2-2\mu-2\sigma)+ \\ \qquad (1+2\lambda)^k(\delta+1)(\delta+2)(3-\mu^2-\mu\sigma-\sigma^2)\end{bmatrix}}.$$

Now applying Lemma 1 to coefficients c_2 and d_2, we get

$$|a_2| \leq \sqrt{\frac{4(1-\beta)}{\begin{matrix}2(1+\lambda)^{2k}(\delta+1)^2|\mu^2+2\mu\sigma+\sigma^2-2\mu-2\sigma|+ \\ \qquad (1+2\lambda)^k(\delta+1)(\delta+2)|3-\mu^2-\mu\sigma-\sigma^2|\end{matrix}}}$$

which is required estimate on $|a_2|$.
Now to find the bound on $|a_3|$, subtracting (3.10) from (3.8), we get

$$\begin{aligned} &2(1+2\lambda)^k\frac{(\delta+1)(\delta+2)}{2}(3-\mu^2-\mu\sigma-\sigma^2)a_3 - (1+2\lambda)^k(\delta+1)(\delta+2) \\ &\quad (3-\mu^2-\mu\sigma-\sigma^2)a_2^2 = (1-\beta)(c_2-d_2). \end{aligned} \tag{3.13}$$

Using (3.12) in (3.13), we get

$$a_3 = \frac{(1-\beta)(c_2-d_2)}{(1+2\lambda)^k(\delta+1)(\delta+2)(3-\mu^2-\mu\sigma-\sigma^2)} + \frac{(1-\beta)^2(c_1^2+d_1^2)}{2(1+\lambda)^{2k}(\delta+1)^2(2-\mu-\sigma)^2}.$$

Applying Lemma 1, we get

$$|a_3| \leq \frac{4(1-\beta)}{(1+2\lambda)^k(\delta+1)(\delta+2)|3-\mu^2-\mu\sigma-\sigma^2|} + \frac{4(1-\beta)^2}{(1+\lambda)^{2k}(\delta+1)^2|(2-\mu-\sigma)^2|}. \tag{3.14}$$

Which is required estimate on $|a_3|$.

Putting $k=0$ in Theorem 3.1, we have the following Corollary.

Corollary 5. *Let $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma^0(\beta,\delta,\mu,\sigma)$. Then*

$$|a_2| \leq \sqrt{\frac{4(1-\beta)}{2(\delta+1)^2|\mu^2+2\mu\sigma+\sigma^2-2\mu-2\sigma| + (\delta+1)(\delta+2)|3-\mu^2-\mu\sigma-\sigma^2|}}$$

and

$$|a_3| \leq \frac{4(1-\beta)}{(\delta+1)(\delta+2)|3-\mu^2-\mu\sigma-\sigma^2|} + \frac{4(1-\beta)^2}{(\delta+1)^2|(2-\mu-\sigma)^2|}.$$

Putting $\delta = 0$ in Corollary 5, we get the following Corollary.

Corollary 6. [2] *Let $f(z)$ given by (1.1) be in the class $\mathcal{H}_{\Sigma}^{0}(\beta,\mu,\sigma)$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{|3+\mu\sigma-2\mu-2\sigma|}} \quad \text{and} \quad |a_3| \leq \frac{2(1-\beta)}{|3-\mu^2-\mu\sigma-\sigma^2|} + \frac{4(1-\beta)^2}{|(2-\mu-\sigma)^2|}.$$

Putting $\mu = 1$ and $\sigma = -1$ in Corollary 6, we get the following Corollary.

Corollary 7. [2] *Let $f(z)$ given by (1.1) be in the class $\mathcal{H}_{\Sigma}^{0}(\beta,1,-1)$. Then*

$$|a_2| \leq \sqrt{1-\beta} \quad \text{and} \quad |a_3| \leq (1-\beta)(2-\beta).$$

Putting $\mu = 1$ and $\sigma = 0$ in Corollary 6, we get the following consequence.

Corollary 8. [13] *Let $f(z)$ given by (1.1) be in the class $S_{\Sigma}^{*}(\beta)$. Then*

$$|a_2| \leq \sqrt{2(1-\beta)} \quad \text{and} \quad |a_3| \leq (1-\beta)(5-4\beta).$$

4. Conclusion

The new subclasses of class of bi-univalent functions in $\mathbb{U}$, $\mathcal{H}_{\Sigma}^{k}(\alpha,\delta,\lambda,\mu,\sigma)$ and $\mathcal{H}_{\Sigma}^{k}(\alpha,\delta,\lambda,\mu,\sigma)$ have been presented by using generalized differential operator and also we have estimated the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in each of these bi-univalent classes. After using particular values for parameters used in our main results we obtain results proved by authors [13, 2, 4] and several other new results were discovered. The results obtained in this paper are not sharp and it is open for others to prove their sharpness, also for $|a_n|, n \geq 4$.

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